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A characterization of Dunkl-classical d-symmetric d-orthogonal polynomials and its applications

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ABSTRACT

In this paper, we obtain a necessary and sufficient condition on a linear operator J, defined on polynomials, and a d-symmetric d-orthogonal polynomial set $\{P_n\}_{n\geq 0}$ such that $\{JP_n\}_{n\geq 0}$ is also d-orthogonal. That allows us to characterize the Dunkl-classical d-symmetric d-orthogonal polynomials as the range of classical d-symmetric d-orthogonal polynomials by an operator related to the Dunkl operator. As applications, we derive many properties of the Dunkl-classical d-symmetric d-orthogonal polynomials from the classical d-symmetric d-orthogonal polynomials.

1. Introduction

The classical orthogonal polynomials sets are the Jacobi, Laguerre, and Hermite polynomials, and all sets which can be derived from them by means of an affine transformation of the variable. Such polynomials have been the object of extensive works. It has been shown that they enjoy a number of similar properties which in turn characterize them. For instance, the following hold.

(a) They all have derivatives which form orthogonal polynomial sets [1].
(b) They all satisfy a second-order linear differential equation of the Sturm–Liouville type [2]: $\sigma(x)y''(x) + \tau(x)y'(x) + \lambda_n y(x) = 0$, where $\sigma(x)$ is a polynomial of degree ≤ 2 and $\tau(x)$ is a polynomial of degree 1, both independent of n, and λ_n is independent of x.
(c) They all possess a Rodrigues-type formula [3]:
$P_n(x) = \frac{1}{K_n w(x)} \frac{d^n}{dx^n}\{w(x)\sigma^n(x)\}$, $n \in \mathbb{N}$, $w(x)$ being a non-negative function on a certain interval.
(d) They are all orthogonal with respect to a weight function $w(x)$ satisfying a Pearson differential equation [4]: $(\sigma(x) w(x))' = \tau(x)w(x)$.
(e) They all satisfy the differentiation formula [5]: $\sigma(x)P_n'(x) = (\alpha_n x + \beta_n)P_n(x) + \gamma_n P_{n-1}(x)$.

The notion of classical orthogonal polynomials has been generalized in two directions.

The first was initiated by Hahn [6], who replaced the derivative operator by the operator

$$L_{q,w}(f)(x) := \frac{f(qx + w) - f(x)}{(q-1)x + w},$$

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where q and w are fixed numbers with $q \neq 1$. He stated some characterization theorems for $L_{q,w}$-classical orthogonal polynomials similar to (a)–(e). He also expressed these polynomials in terms of basic hypergeometric series. Since the derivative operator D is a limiting case of the Hahn operator ($w = 0$ and $q \to 1$ of $L_q := L_{q,0}$), the classical orthogonal polynomials are included in the Hahn class as limiting cases. The operator $L_{1,1}$ is reduced to the finite difference operator Δ given by $\Delta f(x) = f(x+1) - f(x)$. The literature on these topics is extremely vast. We quote, for instance, [7–9,2,1,6,10–12].

The second direction, suggested by the property (a), was initiated in [13], where the authors replaced the notion of orthogonality by the notion of d-orthogonality (see Definition 2.1); i.e., a polynomial set $\{P_n\}_{n\geq 0}$ ($degP_n = n$) is called classical d-orthogonal if and only if both $\{P_n\}_{n\geq 0}$ and $\{P'_{n+1}\}_{n\geq 0}$ are d-orthogonal. Some characteristic properties of the d-symmetric d-orthogonal polynomials have been given by Ben Cheikh and Ben Romdhane [14].

Now, combining these two directions, it is possible to speak about the $L_{q,w}$-classical d-orthogonal polynomials according to the Hahn property. Some examples of Δ-classical d-orthogonal polynomials and L_q-classical d-orthogonal polynomials may be found in [15–19].

In this context, it is also possible to speak about the notion of T_μ-classical d-orthogonal polynomials (or Dunkl-classical d-orthogonal polynomials), where T_μ is the Dunkl operator defined by [20]

$$T_\mu = D + 2\mu L_{-1}, \quad \mu > -1/2,$$

since T_μ is a lowering operator. Such operators has been used recently for various questions related to harmonic analysis. A polynomial set $\{P_n\}_{n\geq 0}$ is then a T_μ-classical d-orthogonal polynomial set if and only if $\{T_\mu P_{n+1}\}_{n\geq 0}$ is also a d-orthogonal polynomial set. This new notion, for $d = 1$, was introduced in [21]. It is in fact significant to investigate d-orthogonal polynomials classified according to lowering operators. Similar investigations were initiated, for $d = 1$, in [22], where the authors raised and solved partially the following question: for which operator L of type

$$L(x^n) = \mu_n x^{n-1}, \quad n = 0, 1, \ldots, \ \mu_0 = 0, \ \mu_n \neq 0 \ (n = 1, 2, \ldots)$$

and for which orthogonal polynomial sets $\{P_n\}_{n\geq 0}$ is the set $\{LP_{n+1}\}_{n\geq 0}$ also d-orthogonal?

In [23], we investigated a subclass of Dunkl-classical d-orthogonal polynomials for odd positive integer d, namely, the Dunkl–Appell polynomials. In this paper, for a positive odd integer d, we characterize the Dunkl-classical d-symmetric d-orthogonal polynomials as the range of classical d-symmetric d-orthogonal polynomials by the operator V_μ given by (2.6).

Our main result is the following.

Theorem 1.1. *Let (μ, d) be in $(]-1/2, +\infty[) \setminus \{0\} \times (2\mathbb{N}+1)$ and let $\{h_n\}_{n\geq 0}$ be a d-symmetric d-orthogonal polynomial set. Then the following statements are equivalent.*

(a) $\{h_n\}_{n\geq 0}$ *is a T_μ-classical d-orthogonal polynomial set.*
(b) $\{V_\mu^{-1} h_n\}_{n\geq 0}$ *is a classical d-orthogonal polynomial set.*

As applications, we derive many interesting results for T_μ-classical d-symmetric d-orthogonal polynomials. The outline of the paper is as follows. In Section 2, we prove our main result. We recall some definitions and results to be used in what follows, we discuss a general problem concerning d-orthogonality-preserving operators, we give a necessary and sufficient condition on the operator J given by (2.12) and d-symmetric polynomial sets $\{P_n\}_{n\geq 0}$ defined by (2.13) to have both $\{P_n\}_{n\geq 0}$ and $\{JP_n\}_{n\geq 0}$$d$-orthogonal, and we apply this result to classical d-symmetric d-orthogonal polynomal sets and the operator V_μ. That, in Section 3, allows us establish some results related to T_μ-classical d-symmetric d-orthogonal polynomials. Finally, in Section 4, by considering an example, we show that the d-orthogonality is not preserved by the operator V_μ for the non-symmetric orthogonal polynomials, and we discuss the possibilities to extend the results obtained.

2. Proof of the main result

Theorem 1.1, for $d = 1$, is reduced to Theorem 3.4 in [21], where we solved a specific T_μ-equation. In this section, we use another approach to state Theorem 1.1, based on d-orthogonality-preserving operators. Next, we recall some useful results.

2.1. Preliminaries and notation

Throughout this paper, we shall use the following notation, definitions and formulas.

2.1.1. Dunkl operator

Let μ be a real number satisfying $\mu > -1/2$. The Dunkl operator T_μ is defined by [20,24]

$$T_\mu \phi(x) = \phi'(x) + \mu \frac{\phi(x) - \phi(-x)}{x}, \tag{2.1}$$

ϕ being an entire function. The operator T_0 is reduced to the derivative operator D.

For non-negative integer n, put

$$\gamma_\mu(2n) = \frac{2^{2n} n!\, \Gamma(n+\mu+1/2)}{\Gamma(\mu+1/2)} = (2n)! \frac{\Gamma(n+\mu+1/2)\Gamma(1/2)}{\Gamma(n+1/2)\Gamma(\mu+1/2)} \tag{2.2}$$

and

$$\gamma_\mu(2n+1) = \frac{2^{2n+1} n!\, \Gamma(n+\mu+3/2)}{\Gamma(\mu+1/2)} = (2n+1)! \frac{\Gamma(n+\mu+3/2)\Gamma(1/2)}{\Gamma(n+3/2)\Gamma(\mu+1/2)}. \tag{2.3}$$

γ_μ plays the role of a generalized factorial [25], since

$$\gamma_\mu(n+1) = (n+1+2\mu\theta_{n+1})\gamma_\mu(n), \quad n \in \mathbb{N} := \{0, 1, 2, \ldots\},$$

where θ_n is defined to be 0 if $n \in 2\mathbb{N}$ and 1 if $n \in (2\mathbb{N}+1)$.

One easily verifies that

$$T_\mu x^n = \frac{\gamma_\mu(n)}{\gamma_\mu(n-1)} x^{n-1}, \quad n \geq 1, \qquad T_\mu(1) = 0. \tag{2.4}$$

It was shown in [20] that the associated commutative algebra of the Dunkl operator T_μ is intertwined with the algebra of the usual derivative operator D by a unique linear and homogeneous isomorphism V_μ on polynomials. More precisely,

$$V_\mu(\mathcal{P}_n) = \mathcal{P}_n, \qquad V_\mu(1) = 1$$

and

$$T_\mu V_\mu = V_\mu D, \tag{2.5}$$

where $\mathcal{P}_n$ denotes the set of polynomials of degree less than or equal to n.

V_μ is given explicitly by

$$V_\mu(x^n) = \frac{\left(\frac{1}{2}\right)_{[\frac{n+1}{2}]}}{\left(\mu+\frac{1}{2}\right)_{[\frac{n+1}{2}]}} x^n = \frac{n!}{\gamma_\mu(n)} x^n, \quad n = 0, 1, \ldots, \tag{2.6}$$

where $[x]$ denotes the greatest integer in x and $(a)_p$ denotes the Pochhammer symbol, given by $(a)_p = \frac{\Gamma(a+p)}{\Gamma(a)}$.

The converse operator of V_μ is given by

$$V_\mu^{-1}(x^n) = \frac{\gamma_\mu(n)}{n!} x^n, \quad n = 0, 1, \ldots. \tag{2.7}$$

From (2.5), we deduce that

$$DV_\mu^{-1} = V_\mu^{-1} T_\mu. \tag{2.8}$$

2.1.2. *d-orthogonal polynomials*

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its algebraic dual. We denote by $\langle u, f\rangle$ the effect of the functional $u \in \mathcal{P}'$ on the polynomial $f \in \mathcal{P}$. A polynomial sequence $\{P_n\}_{n\geq 0}$ is called a *polynomial set* (PS, for short) if and only if $\deg P_n = n$ for all non-negative integer n. The PS $\{P_n\}_{n\geq 0}$ is called monic if $P_n = x^n + \pi_{n-1}$ with deg $\pi_{n-1} \leq n-1$.

Throughout this paper, d denotes a positive integer.

Definition 2.1. A PS $\{P_n\}_{n\geq 0}$ is called *d-orthogonal* (d-OPS, for short) with respect to the d-dimensional functional vector $\Gamma = {}^t(\Gamma_0, \Gamma_1, \ldots, \Gamma_{d-1})$ if it satisfies the following orthogonality relations:

$$\begin{cases} \langle \Gamma_k, P_r P_n\rangle = 0, & r > nd+k, n \in \mathbb{N}, \\ \langle \Gamma_k, P_n P_{nd+k}\rangle \neq 0, & n \in \mathbb{N}, \end{cases} \tag{2.9}$$

for each integer k belonging to $\{0, 1, \ldots, d-1\}$.

For $d = 1$, the d-orthogonality is reduced to the orthogonality (OPS for short).

2.1.3. d-symmetric polynomials

We denote by $w_{d+1} = \exp(2\mathrm{i}\frac{\pi}{d+1})$ the complex $(d+1)$-th root of unity, by $\mathbb{N}_d := \{0, 1, \ldots, d-1\}$ the set of the first d integers, and by $\mathbb{N}_d^*$ the set $\mathbb{N}_d \setminus \{0\}$.

Definition 2.2 (*[26], p. 83*)**.** A PS $\{P_n\}_{n\geq 0}$ is called d-symmetric if it fulfills, for all $n \in \mathbb{N}$,

$$P_n(w_{d+1}x) = w_{d+1}^n P_n(x). \tag{2.10}$$

For $d = 1$, $w_2 = -1$, the PS $\{P_n\}_{n\geq 0}$ is symmetric, i.e., $P_n(-x) = (-1)^n P_n(x)$.

Lemma 2.3 (*[26]*)**.** *Let $\{P_n\}_{n\geq 0}$ be a monic d-OPS. Then, the following statements are equivalent.*

(a) *$\{P_n\}_{n\geq 0}$ is d-symmetric.*
(b) *$\{P_n\}_{n\geq 0}$ satisfies the $(d+1)$-order recurrence relation*

$$\begin{cases} P_n(x) = x^n, & n \in \mathbb{N}_{d+1} \\ P_{n+1} = xP_n(x) - \delta_n P_{n-d}(x), & n \geq d, \end{cases} \tag{2.11}$$

where $\delta_n \neq 0$, $n \geq d$.

2.2. Lemmas

2.2.1. d-orthogonality via transformation

In this section, we shall be concerned with linear operators defined on polynomials by means of

$$Jx^n = \alpha_n x^n, \quad n = 0, 1, 2, \ldots, \tag{2.12}$$

where $\alpha_0 = 1$, $\alpha_n \neq 0$ for all $n > 0$, and with d-symmetric PSs.

Let $\{P_n\}_{n\geq 0}$ be a d-symmetric sequence. P_n is given by its explicit representation,

$$P_n(x) = \sum_{k=0}^{[\frac{n}{d+1}]} c_{n,k} x^{n-(d+1)k}, \tag{2.13}$$

with the condition $c_{n,0} \neq 0$ for all $n \in \mathbb{N}$. In particular, $\{P_n\}_{n\geq 0}$ is monic if and only if $c_{n,0} = 1$.

We raise the following question: for which d-OPSs $\{P_n\}_{n\geq 0}$ defined by (2.13) and operators J of the form (2.12) is the PS $\{Q_n = JP_n\}_{n\geq 0}$ also d-orthogonal?

We state the following.

Lemma 2.4. *Let $\{P_n\}_{n\geq 0}$ be a monic d-symmetric d-OPS given by (2.13). In order that $\{JP_n\}_{n\geq 0}$, where J is defined by (2.12), is also a d-OPS, it is necessary and sufficient that the expression*

$$\gamma_{n,k} = \left(\frac{\alpha_{n-(d+1)k}}{\alpha_{n+1-(d+1)k}} - \frac{\alpha_n}{\alpha_{n+1}}\right)\frac{c_{n,k}}{c_{n-d,k-1}}, \quad n \geq d,\ k = 0, 1, \ldots, \left[\frac{n}{d+1}\right]$$

is independent of k. In this case, we put $\beta_n := \gamma_{n,k}$.

To prove Lemma 2.4, we need the following.

Lemma 2.5. *Let $\{P_n\}_{n\geq 0}$ be a d-symmetric PS given by (2.13). Then $\{P_n\}_{n\geq 0}$ is a d-OPS if and only if there exists a sequence $\{\delta_n\}_{n\geq d}$ such that*

$$c_{n,k} - c_{n+1,k} = \delta_n c_{n-d,k-1}, \tag{2.14}$$

$k \in \{1, 2, \ldots, [\frac{n}{d+1}]\}$.

Proof. According to Lemma 2.3, the sequence $\{P_n\}_{n\geq 0}$ is d-symmetric d-orthogonal if and only if it satisfies the $(d+1)$-order recurrence relation (2.11).

Substituting (2.13) into (2.11), and equating the coefficients of $x^{n-(d+1)k}$, $k \in \{1, 2, , \ldots, [\frac{n}{d+1}]\}$, we obtain the desired result. □

Proof of Lemma 2.4. The sequence $\{P_n\}_{n\geq 0}$ is d-symmetric d-orthogonal. According to Lemma 2.5, the coefficients in the representation (2.13) satisfy the condition (2.14).

On the other hand, since $\{JP_n\}_{n\geq 0}$ is also d-symmetric, $\{JP_n\}_{n\geq 0}$ is a d-OPS if and only if there exists a set $(\epsilon_n)_{n\geq d}$, $\epsilon_n \neq 0\ \forall n \geq d$, such that

$$h_{n+1}(x) = xh_n(x) - \epsilon_n h_{n-d}, \quad n \geq d,$$

where h_n is the monic polynomial defined by $h_n := \frac{1}{\alpha_n} J(P_n)$.

Since the PS $\{h_n\}_{n\geq 0}$ is d-symmetric, we can write h_n, $n \in \mathbb{N}$, under the form (2.13):

$$h_n(x) = \sum_{k=0}^{[\frac{n}{d+1}]} b_{n,k} x^{n-(d+1)k},$$

where the coefficients involved satisfy

$$b_{n,k} = \frac{\alpha_{n-(d+1)k}}{\alpha_n} c_{n,k}. \tag{2.15}$$

According to Lemma 2.5, for $1 \leq k \leq [\frac{n-d}{d+1}]$, $b_{n,k}$ satisfies

$$b_{n,k} - b_{n+1,k} = \epsilon_n b_{n-d,k-1}, \quad n \geq d.$$

We substitute (2.15) in the last equation, taking account of (2.14), to obtain

$$\begin{aligned}
\epsilon_n &= \alpha_{n-d}\left(\frac{\alpha_{n-(d+1)k}}{\alpha_{n+1-(d+1)k}\alpha_n}\frac{c_{n,k}}{c_{n-d,k-1}} - \frac{c_{n+1,k}}{\alpha_{n+1}c_{n-d,k-1}}\right)\\
&= \alpha_{n-d}\left(\frac{\alpha_{n-(d+1)k}}{\alpha_{n+1-(d+1)k}\alpha_n}\frac{c_{n,k}}{c_{n-d,k-1}} - \frac{c_{n,k} - \delta_n c_{n-d,k-1}}{\alpha_{n+1}c_{n-d,k-1}}\right)\\
&= \frac{\alpha_{n-d}}{\alpha_{n+1}}\delta_n + \frac{\alpha_{n-d}}{\alpha_n}\left(\frac{\alpha_{n-(d+1)k}}{\alpha_{n+1-(d+1)k}} - \frac{\alpha_n}{\alpha_{n+1}}\right)\frac{c_{n,k}}{c_{n-d,k-1}}.
\end{aligned} \tag{2.16}$$

The quantity ϵ_n is independent of k if and only if there exists a complex numbers sequence $(\beta_n)_{n\geq d}$ such that

$$\beta_n = \left(\frac{\alpha_{n-(d+1)k}}{\alpha_{n+1-(d+1)k}} - \frac{\alpha_n}{\alpha_{n+1}}\right)\frac{c_{n,k}}{c_{n-d,k-1}}, \quad n \geq d,$$

and, in this case,

$$\epsilon_n = \frac{\alpha_{n-d}}{\alpha_{n+1}}\left(\delta_n + \frac{\alpha_{n+1}}{\alpha_n}\beta_n\right), \quad n \geq d. \quad \square \tag{2.17}$$

Next, we consider the two particular cases where $J = V_\mu$ and $J = V_\mu^{-1}$ given, respectively, by (2.6) and (2.7).

Lemma 2.6. *Let d be an odd positive integer and let $\{P_n\}_{n\geq 0}$ be a monic d-symmetric d-OPS defined by (2.13). Then $\{V_\mu P_n\}_{n\geq 0}$ is a d-OPS if and only if there exists a sequence of complex numbers $\{\tilde{\beta}_n\}_{n\geq d}$ such that*

$$c_{n,k} = \frac{(n+1-(d+1)k)}{(d+1)k}\tilde{\beta}_n c_{n-d,k-1}.$$

Moreover, if $\{P_n\}_{n\geq 0}$ satisfies the $(d+1)$-order recurrence relation (2.11), $\{h_n := \frac{\gamma_\mu(n)}{n!} V_\mu P_n\}_{n\geq 0}$ satisfies the $(d+1)$-order recurrence relation

$$\begin{cases} h_n(x) = x^n, & n \in \mathbb{N}_{d+1} \\ h_{n+1}(x) = xh_n(x) - \epsilon_n h_{n-d}(x), & n \geq d, \end{cases} \tag{2.18}$$

with

$$\epsilon_n = \frac{(n-d)!}{\gamma_\mu(n-d)}\frac{\gamma_\mu(n)}{(n+1)!}((n+1)\delta_n + 2\mu\theta_{n+1}(\delta_n + \tilde{\beta}_n)), \quad n \geq d. \tag{2.19}$$

Proof. Put

$$h_n(x) = \frac{\gamma_\mu(n)}{n!}V_\mu(P_n(x)) = \sum_{k=0}^{[\frac{n}{d+1}]} b_{n,k} x^{n-(d+1)k}.$$

From the identities (2.6) and (2.15), we deduce that $\alpha_n = \frac{n!}{\gamma_\mu(n)}$ and

$$b_{n,k} = \frac{(n-(d+1)k)!}{\gamma_\mu(n-(d+1)k)} \frac{\gamma_\mu(n)}{n!} c_{n,k}. \tag{2.20}$$

On the other hand,

$$\begin{aligned}\frac{\alpha_{n-(d+1)k}}{\alpha_{n+1-(d+1)k}} - \frac{\alpha_n}{\alpha_{n+1}} &= \frac{(n+1-(d+1)k+2\mu\theta_{n+1-(d+1)k})}{(n+1-(d+1)k)} - \frac{(n+1+2\mu\theta_{n+1})}{(n+1)} \\ &= \frac{2\mu(n+1)(\theta_{n+1-(d+1)k}-\theta_{n+1})+2\mu(d+1)k\theta_{n+1}}{(n+1-(d+1)k)(n+1)},\end{aligned}$$

which becomes, if d is odd,

$$\frac{\alpha_{n-(d+1)k}}{\alpha_{n+1-(d+1)k}} - \frac{\alpha_n}{\alpha_{n+1}} = \frac{2\mu\theta_{n+1}}{(n+1)} \frac{(d+1)k}{(n+1-(d+1)k)}.$$

According to Lemma 2.4, the set $\{h_n\}_{n\geq 0}$ is d-orthogonal if and only if there exists a sequence of complex numbers $(\tilde{\beta}_n)_{n\geq d}$ such that

$$\tilde{\beta}_n = \frac{(d+1)k}{(n+1-(d+1)k)} \frac{c_{n,k}}{c_{n-d,k-1}}, \quad n \geq d, \tag{2.21}$$

and, by virtue of (2.17), one obtains (2.19). □

Lemma 2.7. *Let d be an odd positive integer and let $\{P_n\}_{n\geq 0}$ be a monic d-symmetric d-OPS defined by (2.13). Then $\{V_\mu^{-1}P_n\}_{n\geq 0}$ is a d-OPS if and only if there exists a sequence of complex numbers $\{\tilde{\beta}_n\}_{n\geq d}$, $\beta_n \neq 0$, such that*

$$c_{n,k} = \frac{(n+1-(d+1)k+2\mu\theta_{n+1})}{(d+1)k} \tilde{\beta}_n c_{n-d,k-1}, \quad n \geq d.$$

Moreover, if $\{P_n\}_{n\geq 0}$ satisfies the $(d+1)$-order recurrence relation (2.11), $\{V_\mu^{-1}P_n\}_{n\geq 0}$ satisfies the $(d+1)$-order recurrence relation

$$\frac{(n+1)!}{\gamma_\mu(n+1)} V_\mu^{-1}(P_{n+1}(x)) = x\frac{n!}{\gamma_\mu(n)} V_\mu^{-1}(P_n(x)) - \epsilon_n \frac{(n-d)!}{\gamma_\mu(n-d)} V_\mu^{-1}(P_{n-d}(x)), \tag{2.22}$$

with

$$\epsilon_n = \frac{\gamma_\mu(n-d)}{(n-d)!} \frac{n!}{\gamma_\mu(n+1)} ((n+1)\delta_n - 2\mu\theta_{n+1}\tilde{\beta}_n). \tag{2.23}$$

Proof. Since the sequence $\{P_n\}_{n\geq 0}$ is d-symmetric d-orthogonal, according to Lemma 2.5, the coefficients $c_{n,k}$ given by (2.13) satisfy (2.14).

On the other hand, $\{V_\mu^{-1}P_n\}_{n\geq 0}$ is a d-OPS if and only if there exists a sequence of complex numbers $(\epsilon_n)_{n\geq d}$, $\epsilon_n \neq 0$, such that

$$h_{n+1}(x) - xh_n(x) = \epsilon_n h_{n-d},$$

where

$$h_n(x) = \frac{n!}{\gamma_\mu(n)} V_\mu^{-1}(P_n(x)) = \sum_{k=0}^{[\frac{n}{d+1}]} d_{n,k} x^{n-(d+1)k},$$

with

$$d_{n,k} = \frac{\gamma_\mu(n-(d+1)k)}{(n-(d+1)k)!} \frac{n!}{\gamma_\mu(n)} c_{n,k}.$$

According to Lemma 2.5, the coefficients $d_{n,k}$, $1 \leq k \leq [\frac{n-d}{d+1}]$, satisfy

$$d_{n,k} - d_{n+1,k} = \epsilon_n d_{n-d,k-1}.$$

Putting $\alpha_n = \frac{\gamma_\mu(n)}{n!}$, we substitute (2.20) in the last equation, taking account of (2.14), to obtain

$$\begin{aligned}
\epsilon_n &= \frac{\gamma_\mu(n-d)}{(n-d)!}\left(\frac{\alpha_{n-(d+1)k}}{\alpha_{n+1-(d+1)k}}\frac{n!}{\gamma_\mu(n)}\frac{c_{n,k}}{c_{n-d,k-1}} - \frac{(n+1)!}{\gamma_\mu(n+1)}\frac{c_{n+1,k}}{c_{n-d,k-1}}\right)\\
&= \frac{\gamma_\mu(n-d)}{(n-d)!}\left\{\frac{(n+1)!}{\gamma_\mu(n+1)}\delta_n + \left(\frac{\alpha_{n-(d+1)k}}{\alpha_{n+1-(d+1)k}}\frac{n!}{\gamma_\mu(n)} - \frac{(n+1)!}{\gamma_\mu(n+1)}\right)\frac{c_{n,k}}{c_{n-d,k-1}}\right\}\\
&= \frac{\gamma_\mu(n-d)}{(n-d)!}\frac{n!}{\gamma_\mu(n+1)}\left\{(n+1)\delta_n + \left(\frac{\alpha_{n-(d+1)k}}{\alpha_{n+1-(d+1)k}}(n+1+2\mu\theta_{n+1}) - (n+1)\right)\frac{c_{n,k}}{c_{n-d,k-1}}\right\}\\
&= \frac{\gamma_\mu(n-d)}{(n-d)!}\frac{n!}{\gamma_\mu(n+1)}\left\{(n+1)\delta_n + 2\mu\left(\frac{(n+1)(\theta_{n+1}-\theta_{n+1-(d+1)k}) - (d+1)k\theta_{n+1}}{n+1-(d+1)k+2\mu\theta_{n+1-(d+1)k}}\right)\frac{c_{n,k}}{c_{n-d,k-1}}\right\}.
\end{aligned}$$

In particular, if d is odd, we have

$$\epsilon_n = \frac{\gamma_\mu(n-d)}{(n-d)!}\frac{n!}{\gamma_\mu(n+1)}\left\{(n+1)\delta_n - 2\mu\theta_{n+1}\frac{(d+1)k}{n+1-(d+1)k+2\mu\theta_{n+1}}\frac{c_{n,k}}{c_{n-d,k-1}}\right\}.$$

The quantity ϵ_n is independent of k if and only if there exists a sequence of complex numbers $(\tilde{\beta}_n)_{n\geq d}$ such that

$$\tilde{\beta}_n = \frac{(d+1)k}{n+1-(d+1)k+2\mu\theta_{n+1}}\frac{c_{n,k}}{c_{n-d,k-1}}.$$

In this case, ϵ_n is given by (2.23). □

2.2.2. Classical d-symmetric d-orthogonal polynomials

As recalled in the Section 1, Douak and Maroni [13] introduced the notion of classical d-orthogonal polynomials starting from the Hahn property. A polynomial set $\{P_n\}_{n\geq 0}$ is called classical d-orthogonal if and only if both $\{P_n\}_{n\geq 0}$ and $\{P'_{n+1}\}_{n\geq 0}$ are d-orthogonal. In this subsection, we show that the d-symmetric classical d-orthogonal polynomials may be taken as an example of the polynomials described in Lemma 2.6. To this end, we need the following notions and results.

Let $G(z)$ be analytic at $z=0$ with the expansion

$$G(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n \neq 0,\ n\geq 0.$$

Let d be an arbitrary positive integer. Define the monic polynomials P_n, $n = 0, 1, \ldots$, by the generating function

$$G((d+1)xt - t^{d+1}) = \sum_{n=0}^{\infty} c_n P_n(x) t^n, \quad c_n \neq 0,\ n\geq 0. \tag{2.24}$$

Since P_n is monic, we have $c_n = (d+1)^n a_n$, $n \geq 0$.

Ben Cheikh, Douak, and Ben Romdhane stated the following.

Lemma 2.8 ([27,14]). *The only d-orthogonal polynomials generated by (2.24) are the classical d-symmetric polynomials.*

Moreover, if $\{P_n\}_{n\geq 0}$ satisfies the $(d+1)$-order recurrence relation (2.11), then $\{Q_n\}_{n\geq 0}$ satisfies the $(d+1)$-order recurrence relation

$$Q_{n+1}(x) = xQ_n(x) - \widetilde{\delta}_n Q_{n-d}(x), \tag{2.25}$$

where

$$Q_n(x) = (n+1)^{-1}P'_{n+1}(x) \tag{2.26}$$

and

$$\widetilde{\delta}_n = \frac{n+1-d}{n+2}\left[\delta_{n+1} + \frac{c_{n+1-d}}{(n+1)c_{n+1}}\right]. \tag{2.27}$$

Lemma 2.9 ([27]). *The explicit representation for the polynomials P_n generated by (2.24) is given by*

$$P_n(x) = \sum_{k=0}^{[\frac{n}{d+1}]} \frac{(-1)^k (n-dk)!}{(d+1)^k k!(n-(d+1)k)!}\frac{c_{n-dk}}{c_n} x^{n-(d+1)k}.$$

Now, we state the following.

Lemma 2.10. *Let $\{P_n\}_{n\geq 0}$ be a classical d-symmetric d-OPS. Then $\{V_\mu P_n\}_{n\geq 0}$ is a d-OPS.*

Proof. Since $\{P_n\}_{n\geq 0}$ is a classical d-symmetric d-OPS, it is generated by (2.24) according to Lemma 2.8. From the identity (2.13) and Lemma 2.9, we deduce that

$$c_{n,k} = \frac{(-1)^k(n-dk)!}{(d+1)^k k!(n-(d+1)k)!}\frac{c_{n-dk}}{c_n}.$$

This leads to

$$\frac{c_{n,k}}{c_{n-d,k-1}} = -\frac{n+1-(d+1)k}{(d+1)k}\frac{c_{n-d}}{c_n}. \tag{2.28}$$

If we put $\tilde{\beta}_n := \frac{c_{n-d}}{c-n}$ and we use Lemma 2.6, we obtain the desired result. $\square$

2.2.3. Dunkl-classical d-symmetric d-orthogonal polynomials

Replacing the derivative operator in the definition of classical d-OPS, introduced by Douak and Maroni, by the Dunkl operator T_μ, one obtains the following.

Definition 2.11. A PS $\{P_n\}_{n\geq 0}$ is called T_μ-classical (or Dunkl-classical) d-orthogonal if and only if both $\{P_n\}_{n\geq 0}$ and $\{T_\mu P_{n+1}\}_{n\geq 0}$ are d-orthogonal.

Next, using this notion, we give more information for Lemma 2.10.

Lemma 2.12. *Let $\{P_n\}_{n\geq 0}$ be a classical d-symmetric d-OPS. Then $\{V_\mu P_n\}_{n\geq 0}$ is a T_μ-classical d-OPS.*

Proof. According to Lemma 2.8, the polynomials P_n are generated by (2.24). Applying the derivative operator to each member of (2.24) viewed as functions of the variable x, we obtain, by virtue of (2.26),

$$\frac{\partial G}{\partial x}((d+1)xt - t^{d+1}) = \sum_{n=0}^{\infty} c_{n+1}\frac{(n+1)}{d+1}Q_n(x)t^n, \quad c_n \neq 0,\ n\geq 0.$$

Since $\{Q_n\}_{n\geq 0}$ is a d-OPS and is generated by (2.24), then, from Lemmas 2.8 and 2.10, we deduce that $\{V_\mu Q_n\}_{n\geq 0}$ is also d-orthogonal.

On the other hand, from (2.5), we have

$$T_\mu(V_\mu(P_{n+1}(x))) = (V_\mu(P'_{n+1}(x))) = (n+1)(V_\mu(Q_n(x))).$$

This leads to the desired result. $\square$

2.3. Proof of Theorem 1.1

(a) $\Rightarrow$ (b). Let $\{h_n\}_{n\geq 0}$ be a d-OPS which is d-symmetric and T_μ-classical. Consider the three sequences $\{P_n\}_{n\geq 0}$, $\{R_n\}_{n\geq 0}$, and $\{S_n\}_{n\geq 0}$ defined respectively by

$$P_n = V_\mu^{-1}(h_n),$$

$$R_n = \frac{\gamma_\mu(n)}{\gamma_\mu(n+1)}T_\mu(h_{n+1}) := \sum_{k=0}^{[\frac{n}{d+1}]} D_{n,k}x^{n-(d+1)k}, \tag{2.29}$$

and

$$S_n = V_\mu^{-1}(R_n).$$

Put

$$h_n(x) = \sum_{k=0}^{[\frac{n}{d+1}]} b_{n,k}x^{n-(d+1)k}. \tag{2.30}$$

According to Lemma 2.5, for $1\leq k\leq [\frac{n-d}{d+1}]$, the $b_{n,k}$ satisfy

$$b_{n,k} - b_{n+1,k} = \gamma_n b_{n-d,k-1}. \tag{2.31}$$

Since $\{h_n\}_{n\geq 0}$ is a T_μ-classical d-OPS and T_μ takes any d-symmetric PS into a d-symmetric PS, $\{R_n\}_{n\geq 0}$ is also a d-symmetric d-OPS, and hence it satisfies the $(d+1)$-order recurrence relation

$$\begin{cases} R_n(x) = x^n, \quad n\in \mathbb{N}_{d+1}, \\ R_{n+1}(x) = xR_n(x) - \tilde{\gamma}_n R_{n-d}(x), \quad n\geq d. \end{cases}$$

Now, let T_μ operate on both sides of (2.30). Taking into account (2.29) and using (2.4), one obtains

$$D_{n,k} = \frac{\gamma_\mu(n)}{\gamma_\mu(n+1)} b_{n+1,k}(n+1-(d+1)k+2\mu\theta_{n+1-(d+1)k}). \tag{2.32}$$

According to Lemma 2.5, for $1 \le k \le [\frac{n-d}{d+1}]$, the $D_{n,k}$ satisfy

$$D_{n,k} - D_{n+1,k} = \widetilde{\gamma}_n D_{n-d,k-1}. \tag{2.33}$$

Next, we consider the case where d is odd.

Replacing n by $(n+1)$ in (2.31) and substituting (2.32) into (2.33), one obtains

$$\frac{(d+1)k(2\mu(\theta_{n+1}-\theta_{n+2})-1)}{(n+1+2\mu\theta_{n+1})(n+2+2\mu\theta_{n+2})} b_{n+1,k} = \left(\widetilde{\gamma}_n \frac{\gamma_\mu(n-d)}{\gamma_\mu(n+1-d)} - \gamma_{n+1}\right)(n+2-(d+1)k+2\mu\theta_{n+2}) b_{n+1-d,k-1}.$$

This leads to

$$\frac{b_{n+1,k}}{b_{n+1-d,k-1}} = \frac{n+2-(d+1)k+2\mu\theta_{n+2}}{k(d+1)} \beta_{n+1}, \tag{2.34}$$

where

$$\beta_{n+1} = \left(\widetilde{\gamma}_n \frac{\gamma_\mu(n-d)}{\gamma_\mu(n+1-d)} - \gamma_{n+1}\right) \frac{(n+1+2\mu\theta_{n+1})(n+2+2\mu\theta_{n+2})}{(2\mu(\theta_{n+1}-\theta_{n+2})-1)}.$$

According to Lemma 2.7, we deduce that $\{P_n := V_\mu^{-1}(h_n)\}_{n\ge0}$ is also a d-symmetric d-OPS.

Taking account of (2.32) and using (2.34), one obtains

$$\frac{D_{n,k}}{D_{n-d,k-1}} = \frac{(n+1-(d+1)k+2\mu\theta_{n+1})}{k(d+1)} \widetilde{\beta}_n,$$

where $\tilde{\beta}_n = \frac{\gamma_\mu(n)\gamma_\mu(n+1-d)}{\gamma_\mu(n+1)\gamma_\mu(n-d)} \beta_{n+1}$.

Then, by virtue of Lemma 2.7, we deduce that the polynomial set $\{S_n\}_{n\ge0}$ is also a d-symmetric d-OPS. Since

$$D(P_n) = D(V_\mu^{-1}(h_n)) = V_\mu^{-1} T_\mu(h_n) = (n+2\mu\theta_n) V_\mu^{-1}(R_{n-1}) = (n+2\mu\theta_n) S_{n-1},$$

we deduce that $\{P_n\}_{n\ge0}$ is a classical d-OPS.

(b) $\Rightarrow$ (a).

If $\{P_n\}_{n\ge0}$ is a classical d-symmetric d-OPS, by virtue of Lemma 2.12, we deduce that $\{h_n\}_{n\ge0}$ is a T_μ-classical d-symmetric d-OPS. □

3. Applications

In this subsection, we state some results for T_μ-classical d-symmetric d-OPSs. Such results are deduced from results for classical d-symmetric d-OPSs and Theorem 1.1.

3.1. Number of Dunkl-classical d-symmetric d-OPSs

Ben Cheikh and Douak [27] stated that there are 2^d-families of classical d-symmetric d-OPSs. Then, from Theorem 1.1, we deduce the following.

Proposition 3.1. *There are 2^d families of T_μ-classical d-symmetric d-OPSs.*

This result, for $d = 1$, was obtained in [21].

Next, we show how to enumerate these 2^d families by giving their corresponding explicit representations and their $(d+1)$-order recurrence relations.

Recall first that the method used to enumerate the classical d-symmetric d-OPS is based on the recurrence relations (2.11) and (2.25). Ben Cheikh and Douak started by giving the system satisfied by the coefficients δ_n and $\widetilde{\delta}_n$, $n \ge 0$. To solve this system, they put

$$\widetilde{\delta}_n = \delta_{n+1} \frac{n-d+1}{n+1} \vartheta_n, \tag{3.1}$$

and they showed that, for $n \ge 1$ and k fixed ($k \in \mathbb{N}_d$), $(\vartheta_n)_n$ satisfies

$$\vartheta_{dn+k} = \begin{cases} 1 \quad n \ge 0; \\ \text{or} \\ \dfrac{n+\lambda_k+1}{n+\lambda_k}, \quad n \ge 0, \end{cases} \qquad (n \ne 0 \text{ if } k = 0), \tag{3.2}$$

where λ_k are d arbitrary parameters with $\lambda_0 \neq -1, -2, \dots$ and $\lambda_k \neq 0, -1, -2, \dots$ for $k \in \mathbb{N}_d^*$. So, there are 2^d families of classical d-OPSs. Each solution is in fact determined from a d-uplet of the form

$$(\vartheta_{dn}, \vartheta_{dn+1}, \dots, \vartheta_{dn+d-1}). \tag{3.3}$$

The coefficient δ_n in the $(d+1)$-order recurrence relation (2.11) is given by

$$\delta_n = \delta_1 \binom{n}{d} \prod_{j=0}^{n-d-1} \Theta_j, \tag{3.4}$$

where

$$\Theta_j = \begin{cases} \dfrac{j(\vartheta_j - 1) + 1}{(j + d + 2)(\vartheta_{j+1} - 1) + 1}, & j = 1, 2, \dots \\ \dfrac{1}{(d + 2)(\vartheta_1 - 1) + 1}, & j = 0. \end{cases} \tag{3.5}$$

Now, starting from a fixed d-uplet (3.3), if we combine (3.4), (3.5), (2.27), (2.28) and (2.20), we derive the explicit representation of the corresponding PS. The use of (2.18) and (2.19) leads to the corresponding recurrence relation. We illustrate such steps in the two following cases.

Case 1. $\vartheta_n = 1, \quad n \geq 0$.

From (3.5), we easily obtain that $\Theta_n = 1$, $n \geq 0$, and from (3.4) we have

$$\delta_n = \delta_1 \binom{n}{d}, \quad n \geq 0. \tag{3.6}$$

We have, from (2.27) and (3.1),

$$\frac{c_n}{c_{n+d}} = \delta_{n+d}[(n + d + 1)\vartheta_n - (n + d)], \tag{3.7}$$

and from (2.28) we get

$$\frac{c_{n,k}}{c_{n-d,k-1}} = -\frac{n + 1 - (d + 1)k}{(d + 1)k} \delta_1 \binom{n}{d}.$$

Then, with the choice $\delta_1 = \frac{d!}{(d+1)^d}$, we obtain

$$c_{n,k} = \frac{n!(-1)^k}{(d + 1)^{(d+1)k} k!(n - (d + 1)k)!}.$$

This leads to the Gould–Hopper polynomials. From (2.20), the explicit representation of the polynomial $H_n(x) := \frac{\gamma_\mu(n)}{n!} V_\mu(P_n(x))$ is given by

$$H_n(x) = \sum_{k=0}^{[\frac{n}{d+1}]} \frac{(-1)^k \gamma_\mu(n)}{(d + 1)^{(d+1)k} k! \gamma_\mu(n - (d + 1)k)} x^{n-(d+1)k}. \tag{3.8}$$

These polynomials are the Gould–Hopper-type polynomials [23].

From the identities (2.18) and (2.19), we deduce that the Gould–Hopper-type polynomials satisfy the $(d+1)$-recurrence relation

$$H_{n+1}(x) = xH_n(x) - \frac{\gamma_\mu(n)}{\gamma_\mu(n - d)(d + 1)^d} H_{n-d}(x). \tag{3.9}$$

Case 2. Let us now consider the case $\vartheta_{dn+k} \neq 1$, for all indices; that is to say,

$$\vartheta_{dn+k} = \frac{n + \lambda_k + 1}{n + \lambda_k}, \quad n \geq 0,\ k = 0, 1, \dots, d - 1,\ (n \neq 0 \text{ if } k = 0).$$

Here, we are interested in a particular case of the last solution. Indeed, if we put

$$\lambda_k = \frac{(d + 1)\lambda + k}{d}, \quad k = 0, 1, \dots, d - 1,$$

we obtain, when changing the indices $dn + k \to n$,

$$\vartheta_n = \frac{n + (d + 1)\lambda + d}{n + (d + 1)\lambda}, \quad n \geq 1,$$

and we then easily obtain

$$\delta_n = \delta_1 \binom{n}{d} \frac{(n-d+(d+1)\lambda)(\lambda+1)_{n-d-1}}{(\lambda+d+1)_{n-d}}, \quad n \geq 0.$$

With the choice $\delta_1 = \frac{d!\Gamma(d+1)}{(d+1)^{d+1}\Gamma(\lambda+d+1)}$, the last identity can be written as

$$\delta_n = \frac{n!(n-d+(d+1)\lambda)(\lambda)_{n-d}}{(d+1)^{d+1}(n-d)!(\lambda)_{n+1}}, \quad n \geq 0. \tag{3.10}$$

From (3.7) and (2.28), we get

$$c_{n,k} = \frac{n!(-1)^k(\lambda)_{n-dk}}{(d+1)^{(d+1)k}(\lambda)_n k!(n-(d+1)k)!}.$$

This leads to Humbert polynomials. From (2.20), the explicit representation of the polynomial $\mathcal{H}^{\lambda}_{n,d+1}(x) := \frac{\gamma_\mu(n)}{n!} V_\mu(P_n(x))$ is given by

$$\mathcal{H}^{\lambda}_{n,d+1}(x) = \sum_{k=0}^{[\frac{n}{d+1}]} \frac{\gamma_\mu(n)(-1)^k(\lambda)_{n-dk}}{(d+1)^{(d+1)k}(\lambda)_n k!\gamma_\mu(n-(d+1)k)} x^{n-(d+1)k}. \tag{3.11}$$

These polynomials are the generalized Humbert polynomials [28].

From the identity (2.18) and (2.19), we deduce that the generalized Humbert polynomials satisfy the $(d+1)$-order recurrence relation [28]

$$\mathcal{H}^{\lambda}_{n+1,d+1}(x) = x\,\mathcal{H}^{\lambda}_{n,d+1}(x) - \frac{(\lambda)_{n-d}\gamma_\mu(n)(n+(d+1)\lambda-(2\mu+1)d+\mu d\theta_n)}{(d+1)^{d+1}\gamma_\mu(n-d)(\lambda)_{n+1}} \mathcal{H}^{\lambda}_{n-d,d+1}(x).$$

3.2. Dunkl-classical d-symmetric d-orthogonal polynomials of Boas–Buck type

Definition 3.2. A polynomial set $\{P_n\}_{n\geq 0}$ is said to have a generating function of Boas–Buck type if there exists a sequence of nonzero numbers $(c_n)_{n\geq 0}$ such that

$$A(t)B(xC(t)) = \sum_{n=0}^{\infty} c_n P_n(x), \tag{3.12}$$

where $A(t) = \sum_{n=0}^{\infty} a_n t^n$, $B(t) = \sum_{n=0}^{\infty} b_n t^n$, $C(t) = \sum_{n=1}^{\infty} c_n t^n$, and $a_0 b_n c_1 \neq 0\ \forall n \in \mathbb{N}$.

Here, we raise the question of finding all T_μ-classical d-symmetric d-OPSs of Boas–Buck type. We state the following.

Theorem 3.3. *The only T_μ-classical d-symmetric d-OPSs of Boas–Buck type are the Gould–Hopper-type polynomials and the generalized Humbert polynomials.*

This result follows from the two following lemmas.

Lemma 3.4 ([14]). *The only classical d-symmetric d-orthogonal polynomials of Boas–Buck type are the Gould–Hopper polynomials and the Humbert polynomials.*

Lemma 3.5. *A PS $\{P_n\}_{n\geq 0}$ is of Boas–Buck type if and only if $\{V_\mu P_n\}_{n\geq 0}$ is of Boas–Buck type.*

Proof. Letting V_μ operate on both sides of (3.12) viewed as functions of the variable x, we obtain, by virtue of Eq. (2.6),

$$A(t)B_1(xC(t)) = \sum_{n=0}^{\infty} \lambda_n V_\mu P_n(x),$$

where $B_1(t) = \sum_{n=0}^{\infty} \frac{n!}{\gamma_\mu(n)} b_n t^n$. □

Next, we give a relation between the two PSs given by Theorem 3.3.

Proposition 3.6. *The Gould–Hopper-type polynomials given by (3.8) follow from the generalized Humbert polynomials defined by (3.11) by letting $\lambda \to \infty$ in the following way:*

$$\lim_{\lambda\to\infty} \lambda^{\frac{d}{d+1}n} \mathcal{H}^{\lambda}_{n,d+1}\left(\frac{x}{\lambda^{\frac{d}{d+1}}}\right) = H_{n,d+1}(x).$$

Proof. For a fixed $n \in \mathbb{N}$ and $k \in \mathbb{N}_d$, $\lambda^{\frac{d}{d+1}n} \frac{\gamma_\mu(n)(-1)^k(\lambda)_{n-dk}}{(d+1)^{(d+1)k}(\lambda)_n k!\gamma_\mu(n-(d+1)k)} \sim \frac{\gamma_\mu(n)(-1)^k}{(d+1)^{(d+1)k}k!\gamma_\mu(n-(d+1)k)} \lambda^{\frac{d}{d+1}(n-(d+1)k)}$ as $\lambda \to \infty$, and from the identity (3.11) we deduce the desired result. □

For $d = 1$ and $\mu = 0$, the Hermite polynomials $\{H_n := 2^n H_{n,2}\}_{n\geq 0}$ follow from the Gegenbauer polynomials defined by $\{C_n^{\alpha+\frac{1}{2}} := \frac{2^n(\alpha+\frac{1}{2})_n}{n!}\mathcal{H}_{n,2}^{\alpha+\frac{1}{2}}\}_{n\geq 0}$ by letting $\lambda \to \infty$ in the following way (see [29] p. 222, Eq. (9.8.34)):

$$\lim_{\alpha\to\infty} \alpha^{-\frac{1}{2}n} C_n^{\alpha+\frac{1}{2}}\left(\frac{x}{\sqrt{\alpha}}\right) = \frac{H_n(x)}{n!}.$$

3.3. Other characteristic properties

In Section 2, we obtained a characteristic property for T_μ-classical d-symmetric d-OPSs. In this subsection, we use this result to derive two others. To this end, we need the following definitions and results.

Let U_μ be the linear operator defined on polynomials by means of

$$U_\mu(x^n) = \frac{(n+1)\gamma_\mu(n)}{\gamma_\mu(n+1)} x^{n+1}, \tag{3.13}$$

where $\gamma_\mu(n)$ is defined by (2.2) and (2.3).

For an analytic function f, we denote by $(T_\mu)_x f$ (respectively, $(U_\mu)_x f$) the action of T_μ (respectively, U_μ) on f considered as a function of the variable x.

Lemma 3.7 ([14]). *A PS $\{P_n\}_{n\geq 0}$ is a classical d-symmetric d-OPS if and only if there exists a sequence $(c_n)_{n\geq d+1}$, $c_n \neq 0$, such that*

$$\begin{cases} P_n(x) = x^n, \quad n \in \mathbb{N}_{d+1} \\ nc_n P_n + c_{n-d} P'_{n-d}(x) - c_n x P'_n(x) = 0, \quad n \geq d+1. \end{cases} \tag{3.14}$$

We state the following.

Theorem 3.8. *Let $\{h_n\}_{n\geq 0}$ be a d-OPS. Then, the following statements are equivalent.*

(i) *$\{h_n\}_{n\geq 0}$ is a T_μ-classical d-symmetric d-OPS.*

(ii) *$\{h_n\}_{n\geq 0}$ satisfies*

$$\begin{cases} h_n(x) = x^n, \quad n \in \mathbb{N}_{d+1} \\ nc_n h_n + c_{n-d} T_\mu h_{n-d}(x) - c_n U_\mu T_\mu h_n(x) = 0, \quad n \geq d+1. \end{cases} \tag{3.15}$$

(iii) *$\{h_n\}_{n\geq 0}$ is generated by*

$$G_d(x,t) = \sum_{n=0}^{\infty} c_n h_n(x) t^n, \tag{3.16}$$

where $G_d(x,t)$ satisfies

$$(U_\mu)_t (T_\mu)_t G_d - ((U_\mu)_x - t^d)(T_\mu)_x G_d = 0. \tag{3.17}$$

Proof. (i)⇔(ii).

From the identities (2.5) and (3.13), we deduce that $D = V_\mu^{-1} T_\mu V_\mu$ and $U_\mu V_\mu = V_\mu X$.

Then

$$V_\mu(XD) = V_\mu(V_\mu^{-1} U_\mu V_\mu V_\mu^{-1} T_\mu V_\mu) = U_\mu T_\mu V_\mu. \tag{3.18}$$

Let $\{h_n\}_{n\geq 0}$ be a T_μ-classical d-symmetric d-OPS. Then $\{P_n := V_\mu^{-1} h_n\}_{n\geq 0}$ is a classical d-symmetric d-OPS. According to Lemma 3.7, $\{P_n\}_{n\geq 0}$ satisfies (3.14). Applying V_μ to each member of (3.14), taking account of (2.5) and (3.18), one obtains

$$nc_n h_n + c_{n-d} T_\mu h_{n-d}(x) - c_n U_\mu T_\mu h_n(x) = 0, \quad n \geq d+1.$$

In the same way, we apply V_μ^{-1} to each member of (3.15) to prove that (ii)⇒(i).

(ii)⇔(iii).

Let $\{h_n\}_{n\geq 0}$ be a d-OPS generated by (3.16). From (3.16), we have

$$(U_\mu)_t (T_\mu)_t G_d(x,t) = \sum_{n=1}^{\infty} nc_n h_n(x) t^n, \tag{3.19}$$

$$t^d (T_\mu)_x G_d(x,t) = \sum_{n=1}^{\infty} c_n T_\mu h_n(x) t^{n+d} = \sum_{n=d+1}^{\infty} c_{n-d} T_\mu h_{n-d}(x) t^n, \tag{3.20}$$

and

$$(U_\mu)_x(T_\mu)_x G_d(x,t) = \sum_{n=1}^{\infty} c_n U_\mu T_\mu(h_n(x))t^n. \tag{3.21}$$

Substituting (3.19)–(3.21) into (3.16), we obtain

$$\begin{aligned}((U_\mu)_t(T_\mu)_t - ((U_\mu)_x - t^d)(T_\mu)_x)G_d(x,t) &= \sum_{n=d+1}^{\infty}(nc_nh_n(x) - c_nU_\mu T_\mu(h_n(x)) + c_{n-d}T_\mu h_{n-d}(x))t^n \\ &\quad + \sum_{n=1}^{d} c_n(nh_n(x) - U_\mu T_\mu(h_n(x)))t^n.\end{aligned}$$

Since $U_\mu T_\mu x^n = nx^n$ and $h_n(x) = x^n$ for $n \in \mathbb{N}_d$, we see that $\{h_n\}_{n\geq 0}$ satisfies (3.15) if and only if G_d satisfies (3.17). □

For $d = 1$, Theorem 3.8 is reduced to the following.

Corollary 3.9. *Let $\{h_n\}_{n\geq 0}$ be an OPS. Then, the following statements are equivalent.*

(i) *$\{h_n\}_{n\geq 0}$ is a T_μ-classical symmetric OPS.*
(ii) *$\{h_n\}_{n\geq 0}$ satisfies*

$$\begin{cases} h_n(x) = x^n, & n \in \mathbb{N}_2 \\ nc_nh_n + c_{n-1}T_\mu h_{n-1}(x) - c_nU_\mu T_\mu h_n(x) = 0, & n \geq 2. \end{cases} \tag{3.22}$$

(iii) *$\{h_n\}_{n\geq 0}$ is generated by*

$$G_1(x,t) = \sum_{n=0}^{\infty} c_nh_n(x)t^n, \tag{3.23}$$

where $G_1(x,t)$ satisfies

$$(U_\mu)_t(T_\mu)_tG_1 - ((U_\mu)_x - t)(T_\mu)_xG_1 = 0.$$

Notice here that it was shown in [21] that the only T_μ-classical symmetric OPSs are the generalized Hermite polynomials and the generalized Gegenbauer polynomials.

3.4. A differential-difference equation (a T_μ-equation)

In this subsection, we state a differential-difference equation satisfied by T_μ-classical d-symmetric d-orthogonal polynomials using Theorem 1.1 and the following lemma.

Lemma 3.10 *([27]). Let $\{P_n\}_{n\geq 0}$ be a PS generated by (2.24). If $\{P_n\}_{n\geq 0}$ is a d-OPS, then the polynomials P_n, $n = 0, 1, \ldots$, satisfy the following $(d+1)$-order differential equation:*

$$L(y) := \left[D^{d+1} - \frac{c_n}{c_{n-d}}(xD - n)\prod_{j=0}^{d-1}(A_{n-1-j}(xD - n + d + 1) + n - j)\right]y = 0, \quad n > d, \tag{3.24}$$

where

$$A_n = 1 - \frac{c_n}{c_{n-d}}\delta_n = \frac{(n+1)(\vartheta_{n-d} - 1)}{(n+1)(\vartheta_{n-d} - 1) + 1}, \quad n > d, \tag{3.25}$$

and c_n, δ_n, and ϑ_n are the three sequences associated to $\{P_n\}_{n\geq 0}$ respectively defined by (2.24), (2.11) and (3.2).

We have the following.

Theorem 3.11. *Let $\{h_n\}_{n\geq 0}$ be a T_μ-classical d-symmetric d-OPS. The polynomials h_n, $n = 0, 1, \ldots$, satisfy the following $(d+1)$-order differential-difference equation:*

$$\left[T_\mu^{d+1} - \frac{c_n}{c_{n-d}}(U_\mu T_\mu - n)\prod_{j=0}^{d-1}(A_{n-1-j}(U_\mu T_\mu - n + d + 1) + n - j)\right]y = 0, \quad n > d, \tag{3.26}$$

where A_n is given by (3.25), and c_n, δ_n, and ϑ_n are the three sequences associated to $\{V_\mu^{-1}h_n\}_{n\geq 0}$ defined by (2.24), (2.11) and (3.2).

Proof. From (3.18), we deduce that

$$V_\mu (XD)^k = (U_\mu T_\mu)^k V_\mu, \quad k = 0, 1, \ldots. \tag{3.27}$$

Let $\{h_n\}_{n\geq 0}$ be a T_μ-classical d-symmetric d-OPS. This means that the polynomials $\{P_n := V_\mu^{-1} h_n\}_{n\geq 0}$ are classical d-symmetric d-OPS. So, according to Lemmas 2.8 and 3.10, the polynomials $P_n, n \geq 0$ satisfy the $(d+1)$-order differential equation (3.24).

Put

$$L = D^{d+1} + \sum_{k=0}^{d} \alpha_{n,k} (XD)^k. \tag{3.28}$$

Letting V_μ operate on both sides of (3.28), and using (3.27), we deduce that

$$V_\mu L(P_n) = \widetilde{L}(V_\mu(P_n)) = \widetilde{L}(h_n),$$

with

$$\widetilde{L} := T_\mu^{d+1} + \sum_{k=0}^{d} \alpha_{n,k} (U_\mu T_\mu)^k.$$

Since, from Lemma 3.10, we have $L(P_n) = 0$, then $\widetilde{L}(V_\mu(P_n)) = 0$, from which we deduce (3.26). $\square$

Next, we apply Theorem 3.11 to the two families obtained in Theorem 3.3.

1. *Gould–Hopper-type polynomials.*

We have, from (3.6) and (3.7),

$$\frac{c_{n-d}}{c_n} = \delta_n = \frac{n!}{(d+1)^d (n-d)!}, \quad n \geq 0,$$

and then, from (3.25), we have

$$A_n = 1 - \frac{c_n}{c_{n-d}} \delta_{n-d+1} = \frac{(n+1)(\vartheta_{n-d} - 1)}{(n+1)(\vartheta_{n-d} - 1) + 1} = 0, \quad n > d.$$

The differential-difference equation (3.26) becomes [23]

$$(T_\mu^{d+1} - (d+1)^d (U_\mu T_\mu - n))y = 0, \ \ n > d. \tag{3.29}$$

Fixing $d = 1$, in (3.29), we obtain the second-order differential-difference equation satisfied by the generalized Hermite polynomials $\{H_n^\mu\}_{n\geq 0}$. Indeed, for this case, we have

$$(T_\mu^2 - 2(U_\mu T_\mu - n))y = 0, \quad n > 1.$$

Taking into account that, for $k = 0, 1, \ldots, n$ and $n - k \in 2\mathbb{N}$,

$$(T_\mu^2 - 2(U_\mu T_\mu - n))x^k = \left(T_\mu^2 - 2xT_\mu + 2\frac{\gamma_\mu(n)}{\gamma_\mu(n-1)} \right) x^k,$$

the symmetric polynomial $H_n^\mu, n \geq 0$, satisfies the differential-difference equation [21]

$$\left(T_\mu^2 - 2xT_\mu + 2\frac{\gamma_\mu(n)}{\gamma_\mu(n-1)} \right) H_n^\mu(x) = 0.$$

2. *Humbert-type polynomials.*

We have, from (3.7) and (3.10),

$$\frac{c_{n-d}}{c_n} = \frac{n!(\lambda)_{n-d}}{(d+1)^d (n-d)!(\lambda)_n}, \quad n \geq 0,$$

and then, from (3.25), we have

$$A_n = 1 - \frac{c_n}{c_{n-d}} \delta_{n-d+1} = \frac{(n+1)(\vartheta_{n-d} - 1)}{(n+1)(\vartheta_{n-d} - 1) + 1} = \frac{(n+1)d}{(d+1)(n+\lambda)}, \quad n > d.$$

The differential-difference equation (3.26) becomes (3.30) [28].

$$\left(T_\mu^{d+1} - (U_\mu T_\mu - n) \prod_{j=0}^{d-1} (d(U_\mu T_\mu - n + d + 1) + (d+1)(\lambda + n - j - 1)) \right) y = 0, \quad n > d. \tag{3.30}$$

Fixing $d = 1$, in (3.30), we obtain the second-order differential-difference equation satisfied by the generalized Gegenbauer polynomials $\{S_n^{\alpha,\mu-1/2}\}_{n\geq 0}$. Indeed, for this case, we have

$$(T_\mu^2 - (U_\mu T_\mu - n)(U_\mu T_\mu + n + 2\lambda))y = 0, \quad n > 1.$$

Taking into account that, for $k = 0, 1, \ldots, n$ and $n - k \in 2\mathbb{N}$,

$$(n - U_\mu T_\mu)(U_\mu T_\mu + n + 2\lambda)x^k = \left(-x^2 T_\mu^2 - 2(\alpha+1)T_\mu + \frac{\gamma_\mu(n)}{\gamma_\mu(n-1)}\left(\frac{\gamma_\mu(n-1)}{\gamma_\mu(n-2)} + 2(\alpha+1)\right)\right)x^k,$$

with $\alpha = \lambda - \mu - 1/2$, the symmetric polynomial $S_n^{\alpha,\mu-1/2}$, $n \geq 0$, satisfies the differential-difference equation [21]

$$\left((1-x^2)T_\mu^2 - 2(\alpha+1)T_\mu + \frac{\gamma_\mu(n)}{\gamma_\mu(n-1)}\left(\frac{\gamma_\mu(n-1)}{\gamma_\mu(n-2)} + 2(\alpha+1)\right)\right)S_n^{\alpha,\mu-1/2}(x) = 0.$$

3.5. About the T_μ-derivatives

Let $\{P_n\}_{n\geq 0}$ be a PS. The PS $\{T_\mu P_n\}_{n\geq 1}$ is called the T_μ-derivative of $\{P_n\}_{n\geq 0}$.
Here we raise the following question.

If the PS $\{P_n\}_{n\geq 0}$ is a T_μ-classical d-OPS, what about its T_μ-derivative $\{T_\mu P_n\}_{n\geq 1}$?

In this subsection, we provide an answer to this question in the d-symmetry case using the following result derived by Ben Cheikh and Ben Romdhane.

Lemma 3.12 ([14]). *If $\{P_n\}_{n\geq 0}$ is a classical d-symmetric d-OPS, then its derivative $\{P_n'\}_{n\geq 1}$ is also a classical d-symmetric d-OPS.*

We state the following.

Theorem 3.13. *If $\{h_n\}_{n\geq 0}$ is a T_μ-classical d-symmetric d-OPS, the sequence $\{T_\mu^r h_n\}_{n\geq r}$ is T_μ-classical for any non-negative integer r.*

Proof. According to Theorem 1.1, the sequence $\{P_n := V_\mu^{-1} h_n\}_{n\geq 0}$ is a classical d-symmetric d-OPS. From Lemma 3.12, we deduce that $\{DP_n\}_{n\geq 1}$ is also a classical d-symmetric d-OPS; then, $\{V_\mu DP_{n+1}\}_{n\geq 0}$ is a T_μ-classical d-symmetric d-OPS, and from (2.5) we deduce that $\{T_\mu h_n\}_{n\geq 1}$ is a T_μ-classical d-symmetric d-OPS. Therefore, the result is valid for $r = 1$. By induction, we obtain the general case. □

4. Concluding remarks

To end this paper, we discuss the possibilities of extending the results obtained.

1. We have shown that, if a d-symmetric d-OPS $\{P_n\}_{n\geq 0}$ is classical, then $\{V_\mu P_n\}_{n\geq 0}$ is a d-OPS (Lemma 2.10). This suggests a natural extension omitting the d-symmetry property; more precisely, it suggests the following question.

   If $\{P_n\}_{n\geq 0}$ is a classical d-OPS, is $\{V_\mu P_n\}_{n\geq 0}$ a d-OPS?

   The answer to this question is negative since, for $d = 1$, the Laguerre polynomials $\{L_n^\alpha\}_{n\geq 0}$ are orthogonal and classical but $\{V_\mu L_n^\alpha\}_{n\geq 0}$ is not an OPS. To justify this affirmation, we recall first that the Laguerre polynomials are given by

$$L_n^\alpha(x) = \frac{(1+\alpha)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k}{(1+\alpha)_k k!}x^k, \quad \alpha \neq 0, -1, -2, \ldots, \tag{4.1}$$

   and the corresponding inversion formula is given by [30, p. 675]

$$x^n = \sum_{j=0}^{n}\frac{(-n)_j(1+\alpha)_n}{(1+\alpha)_j}L_j^\alpha(x). \tag{4.2}$$

   Letting V_μ operate on both sides of (4.2), and using (2.6), we deduce that the inversion formula related to the PS $\{h_n^\alpha := V_\mu L_n^\alpha\}_{n\geq 0}$ is given by

$$x^n = \frac{(1+\alpha)_n\gamma_\mu(n)}{n!}\sum_{j=0}^{n}\frac{(-n)_j}{(1+\alpha)_j}h_j^\alpha(x). \tag{4.3}$$

   According to the definition of a linear functional vector, we have

$$\langle u_0, x^n\rangle = \frac{(1+\alpha)_n\gamma_\mu(n)}{n!}\sum_{j=0}^{n}\frac{(-n)_j}{(1+\alpha)_j}\langle u_0,\ h_j^\alpha(x)\rangle = \frac{(1+\alpha)_n\gamma_\mu(n)}{n!}. \tag{4.4}$$

On the other hand, applying V_μ to each member of (4.1), for $n = 1, 2$, we deduce that

$$h_1^\alpha(x)h_2^\alpha(x) = -\frac{1}{2(1+2\mu)^2}x^3 + \left(\frac{1+\alpha}{2(1+2\mu)} + \frac{2+\alpha}{(1+2\mu)^2}\right)x^2 - \frac{3(1+\alpha)(2+\alpha)}{2(1+2\mu)}x + \frac{(1+\alpha)^2(2+\alpha)}{2}. \quad (4.5)$$

Applying u_0 to each member of (4.5), taking into account (4.4), for $n = 0, 1, 2, 3$, we deduce that $\langle u_0, h_1^\alpha(x)h_2^\alpha(x)\rangle = -\frac{4\alpha\mu(1+\alpha)(2+\alpha)}{3(1+2\mu)} \neq 0$.

The desired result follows from the following lemma.

Lemma 4.1 ([13]:). *If a PS $\{P_n\}_{n\geq 0}$ is d-orthogonal with respect to a d-dimensional functional vector $\Gamma = {}^t(\Gamma_0, \Gamma_1, \dots, \Gamma_{d-1})$, then this PS is also d-orthogonal with respect to the d-dimensional functional vector $\mathcal{U} = {}^t(u_0, u_1, \dots, u_{d-1})$, where the forms $u_0, u_1, \dots, u_{d-1}$ are the d first elements of the dual sequence $\{u_n\}_{n\geq 0}$ associated to $\{P_n\}_{n\geq 0}$.*

2. In this paper, we gave, in Lemma 2.4, a necessary and sufficient condition on operator J and the d-symmetric d-OPS $\{P_n\}_{n\geq 0}$ to have also $\{JP_n\}_{n\geq 0}$$d$-orthogonal. Then, we fixed an operator J, namely V_μ, to obtain in Lemma 2.6 conditions on the PS $\{P_n\}_{n\geq 0}$ to have both $\{P_n\}_{n\geq 0}$ and $\{V_\mu P_n\}_{n\geq 0}$$d$-orthogonal. Therefore, in Lemma 2.10, we gave an example for Lemma 2.6: a classical d-symmetric OPS. Here, two natural question arise.

   Question 1. Are there other examples for Lemma 2.6?

   Question 2. State an analogue of Lemma 2.6 by fixing the PS as a classical d-symmetric OPS and finding conditions on the operator J. This may be deduced from Lemma 2.4. In this case, are there known operators J different from V_μ?
3. We stated in Theorem 3.11 that the T_μ-classical d-symmetric d-orthogonal polynomials satisfy a differential-difference equation of type (3.26). It is of interest to study the converse in order to derive another characteristic property for T_μ-classical d-symmetric d-orthogonal polynomials similar to Bochner differential equation (see property (b) in Section 1).

4. In this paper, we have limited ourselves to odd positive integer d. The question for general positive integer d remains open.

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